# Monte Carlo results in time-dependent hierarchical fiber-bundle models of fracture

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Is there any time threshold of stability in a time-dependent hierarchical load-transfer structure formed by $N$ elements, in the limit $N\to\infty$? There is no rigorous proof, but the consensus to this question is yes. Here we extend our previous work on these systems up to a size $N=10^6$, using a power law breakdown rule and a new more efficient Monte Carlo method. The new results confirm this positive consensus.

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## I. INTRODUCTION

The modeling of fracture in disordered systems is a subject of great interest in natural and artificial materials [1]. A time-dependent method to describe the failure of materials under stress, within the fiber-bundle paradigm, was proposed by Coleman [2]. In this model, a set (*bundle*) of elements (*fibers*) is considered with each element having a prescribed lifetime when subject to an applied stress (*load*). When elements fail, their load is redistributed to other elements of the set according to a prescribed rule of transfer. As a consequence of the load transfer, the lifetime of the receptors is reduced and the main question is: How long does it take for the whole set to collapse? These fiber-bundle models are called dynamical or time-dependent [3–5], as opposed to their static counterparts, which have also been intensively studied [3,6].

Generally, three rules for the redistribution of the load are considered. In the first, the stress of the failed element is transferred equally to all surviving elements (ELS, for equal load sharing). In the second, the load of the failed element is transferred to the nearest surviving element(s) (LLS, for local load sharing). Hierarchically organized load transfer rules (HLS) are considered in the geophysical literature and especially in seismology [4,7–9]. In this field, the bundle is a simplified representation of a fault, and the individual elements represent asperities in the fault plane. Here, the stress released by a failed sub-bundle is transferred to a neighboring sub-bundle at the same hierarchical level. In other words, the zone of stress transfer is equal in size to the zone of failure, simulating the role of the Green's function associated with the elastic redistribution of stress adjacent to a rupture. These hierarchical arrays have been used to model the precursory seismic activation that occurs before many major earthquakes, and to implement the idea of discrete scale invariance [4,7].

In this paper, we will extend our previous numerical calculations on time-dependent HLS systems [5,10] in order to give more support and confidence to the current status of opinion that these systems have a finite time threshold of stability, in the limit $N\to\infty$. We will consider a hierarchical structure, or fractal tree, with an homogenous degree of branching $c=2$, so that $N=2^n$, $n$ being the number of levels in the binary tree. In Sec. II we review the standard Monte Carlo approach used for the computation of these systems. In Sec. III we present a new Monte Carlo strategy and discuss its practical advantages. Finally in Sec. IV, the new results are graphically shown and we present our conclusions.

## II. STANDARD MONTE CARLO APPROACH

In the standard Monte Carlo approach [3,4] the population of initial lifetimes is fixed at the beginning of the simulation and the subsequent breaking process is deterministic. In other words, the disorder is *quenched*: A random lifetime is assigned to each individual element at the onset, drawn from a probability distribution. This is the only time during the whole simulation where random numbers are utilized.

For engineering and heterogeneous natural materials, failure is often modeled by means of a Weibull distribution,

$$F(t_{i0}) = 1 - \exp[-\{\nu(\sigma)t_{i0}\}^{\beta}], \tag{1}$$

where $\nu(\sigma)$ is the hazard rate under laod $\sigma$ and $\beta$ is the so-called Weibull index. In what follows we will use $\beta=1$, which converts the Weibull distribution in an exponential distribution. Operationally, for a fiber-bundle of size $N$, we draw $N$ random numbers uniformly distributed in the interval [0, 1], which are identified with $F(t_{i0})$, and compute from Eq. (1) a failure time $t_{i0}$ for an initial applied load $\sigma_0$. This initial load is equal for all elements in the bundle.

For the hazard rate $\nu(\sigma)$, two basic modalities are used in the literature [3]: The exponential and the power-law types. Here we will use as example the *power law* hazard rate, but the exponential type can be implemented in the same way. The power-law hazard rate has the form

$$\nu = \nu_0\left(\frac{\sigma}{\sigma_0}\right)^{\rho}, \tag{2}$$

where obviously $\nu_0$ is the hazard rate of a single element under load $\sigma_0$. Without losing generality one can choose $\nu_0=\sigma_0=1$. The exponent $\rho$, called in the rock physics literature the stress corrosion index, is typically in the range between 2 and 5 for manmade structures [4] and 10–50 for rocks [11]. Inserting Eq. (2) into Eq. (1) with $\beta=1$ an exponential distribution for failure times is obtained.

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Due to the load transfer from broken to unbroken elements, the individual load supported by the elements grows during the progressive breakdown of the bundle, thus reducing their time to failure. This reduced time to failure $T_{if}$ is given by [4]

$$t_{i0} = \int_0^{T_{if}} \left( \frac{\sigma(t)}{\sigma_0} \right)^{\rho} dt, \tag{3}$$

where the power-law hazard rate has been used. As commented on above, for solving the problem by a Monte Carlo method each element $i$ in the system is initially assigned a random time to failure $t_{i0}$ under load $\sigma_0$ based on Eq. (1). The actual time to failure of element $i$, $T_{if}$, will be reduced below $t_{i0}$ every time load is transferred to this element due to the failure of other elements in the system. This reduced failure time $T_{if}$ is obtained by requiring that Eq. (3) is satisfied under the new load $\sigma(t)$. The $T_{if}$ corresponding to the last failing element is the lifetime of the bundle, $T$.

### III. THE NEW MONTE CARLO METHOD

In Ref. [2], Coleman formulated the time-dependent breaking of an ELS bundle by means of a differential equation of the radioactive decay type. This continuous formulation cannot be applied to other load transfer schemes but it provides a lucid perspective that leads to a different approach to solve these discrete systems. In Refs. [5,10] this approach was used together with a renormalization procedure to describe exactly the time to failure of a bundle of $2N$ fibers using the values of the times to failure of all the configurations of a bundle with $N$ fibers. Unfortunately though, using this method we could not provide exact results beyond a modest $N=32$ fiber bundle. However, we were able to define upper and lower bounds for the time of collapse, $T$, of large bundles. These bounds correctly forked the Monte Carlo results obtained using the standard approach explained in Sec. II. These Monte Carlo results covered bundle sizes up to $2^{14}=16\,384$ fibers. The results presented in this paper, obtained with a Monte Carlo strategy based on the "radioactive" perspective mentioned above, extend these results up to $2^{20}=1\,048\,576$ fibers. The new perspective is based on a well-known property of the exponential distribution for failure times: The minimum among independent exponentially distributed random variables is also exponentially distributed and its parameter is the sum of parameters of the original variables.

Initially, at $t=0$, the hierarchical bundle—or fractal tree—is intact and all the $N$ elements support a load $\sigma=\sigma_0=1$. The breakdown of the bundle proceeds sequentially, element after element, with an individual breaking *and* its appropriate load redistribution, from $i=1$ until the breaking of the $N$th element, $i=N$. Thus, the $i$ index will denote the successive $N$ steps of breaking of the bundle. The point then is to determine how long it takes to break one element in the $i$th step. This will be denoted by $\delta_i$ and its computation will be specified later. In consequence, the time of collapse of the total bundle is

$$T = \sum_{i=1}^{N} \delta_i. \tag{4}$$

In an arbitrary intermediate state $i$ of breaking, the bundle will be in a configuration with $i$ elements broken, and $N-i$ unbroken bearing different individual loads $\sigma_j$. The $j$ index extends to all $N-i$ unbroken elements of the bundle.

Note that due to the conservation of the total load during the breaking and redistributing processes, at any $i$, $N\sigma_0=N=\Sigma_{j=1}^{N-i}\sigma_j$.

Now, in analogy with a radioactive process, we define the *decay width* of an unbroken element as

$$\Gamma_j = \sigma_j^{\rho}. \tag{5}$$

Note that $\Gamma_j$ is the same thing as $\nu$ in Eq. (2). The total decay width of the configuration at step $i$ can then be written as

$$\Gamma(i) = \sum_{j=1}^{N-i} \Gamma_j. \tag{6}$$

The lifetime of that configuration is simply the inverse of the decay width:

$$\delta_i = \frac{1}{\Gamma(i)}. \tag{7}$$

Finally, the probability of breaking the unbroken element $j$ is given by

$$p_j(i) = \frac{\Gamma_j}{\Gamma(i)}, \tag{8}$$

which is the branching ratio of the $j$th path. Thus, the choice of the specific fiber that breaks in each time step is materialized by generating a random number between 0 and 1. We see that using this approach we will obtain, in principle, different values for the $N$ deltas in each Monte Carlo simulation of the total breaking of a bundle. The mean value of a sufficiently large number of simulations provides a convergent result for $T$, the lifetime of the system. Thus, in contrast to the standard method, here one starts with a unique configuration, and the fluctuations in $T$ arise from the different paths of rupture resulting in each simulation (*annealed disorder*).

Computing times are very different for the standard and new Monte Carlo methods. This speeding up of the algorithm has enabled us to go beyond the limit of $N=2^{14}$ fibers reached in Ref. [10], up to the value $N=2^{20}$ fibers, a factor of 64 in system size. As an example, the time needed to break $10^6$ systems of size $N=2^{10}$ is 29.51 h with the standard method and 6.52 h with the new method (Pentium 4 at 2.40 GHz). An additional benefit of the new Monte Carlo method is a reduction in the dispersion of lifetimes. The lifetimes are normally distributed around the mean lifetime for each system size and the standard deviation is smaller for the new Monte Carlo method. As an example, for an $N=2^{10}$ system the standard deviation around the mean lifetime is 0.016 for the standard method and 0.008 for the new method. This means that less realizations are needed to arrive at a specific maximum error in the mean. Both advantages,

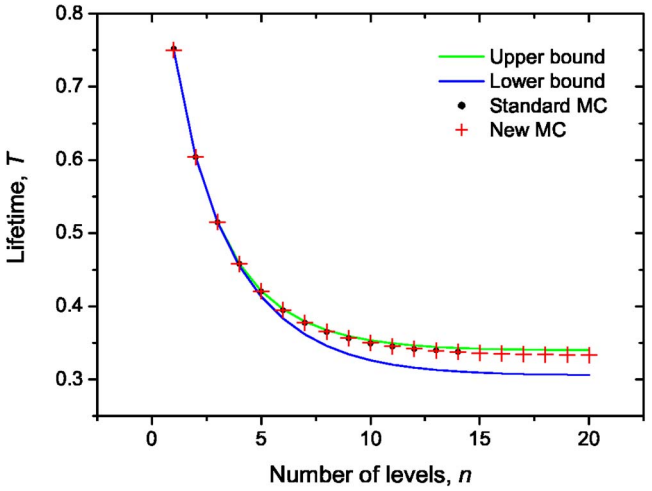


FIG. 1. (Color online) Dimensionless lifetime, $T$, for an HLS binary tree with $n$ levels. Filled circles were calculated with the standard Monte Carlo method. Red crosses have been computed using the new method. The maximum system size with the new method is $N=2^{20}=1\,048\,576$ elements.

the speed and the low deviation, make the new Monte Carlo method a preferred choice when computing large hierarchical time-dependent fiber-bundle systems.

## IV. RESULTS AND DISCUSSION

Figures 1–3 try to demonstrate that HLS systems have indeed a nonzero lifetime in the limit of very large systems. The Monte Carlo data as computed by the new method outlined above are shown in Fig. 1 as red crosses. The number

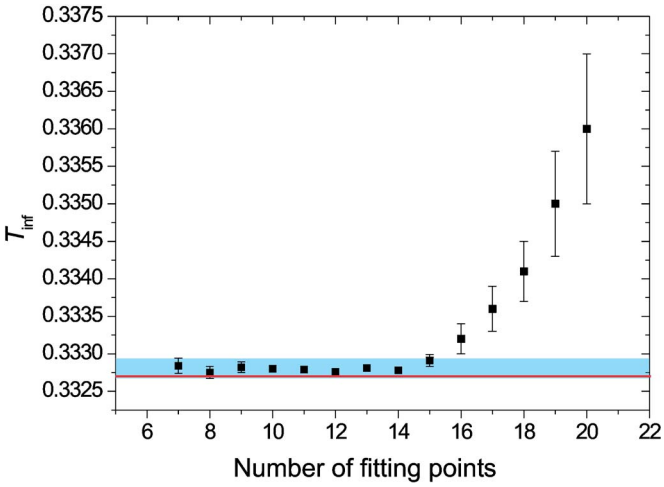


FIG. 2. (Color online) Result of fitting an exponential function to the Monte Carlo data as the number of data points included in the fit is changed from 20 (all data points) to 7 (data points corresponding to the seven biggest systems). After the smaller systems are eliminated from the fit and only the 14 biggest systems are included, the predicted $T_\infty$ stabilizes at a value of $0.3328\pm0.0013$ (shaded area). This range is compatible with the value of 0.3327 obtained from Fig. 3 (red line near the lower limit of the shaded area).

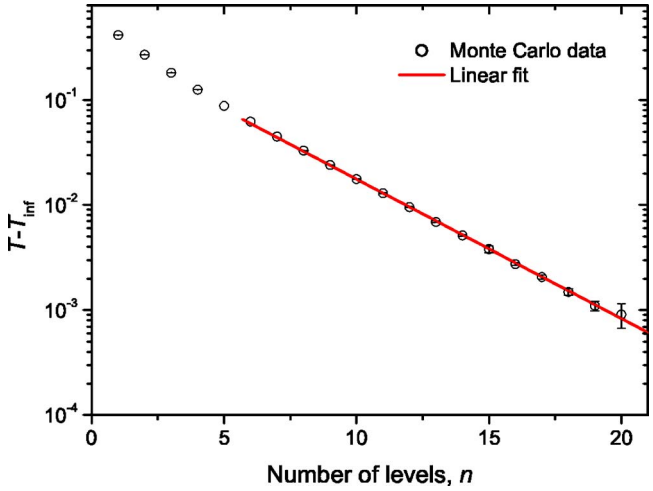


FIG. 3. (Color online) Log-linear plot of Monte Carlo data after subtracting the asymptotic value. The red line is a linear fit from $n=14$ to $n=20$ to show that lifetime data can be accurately approximated by an exponential function. The Monte Carlo points are accompanied by error bars, smaller than the symbol itself in most cases except for $n=20$. The best straight line (linear fit with the highest correlation coefficient, $r=-0.999\,73$) is obtained by subtracting $T_\infty=0.3327$ from the Monte Carlo results.

of realizations for each Monte Carlo point is not constant and goes from $10^7$ realizations for $n<10$ systems to $10^2$ for the biggest system ($n=20$). In order to obtain the lifetime of the system for $N\to\infty$, the Monte Carlo data points were fitted to an exponential function of the form

$$T = T_\infty + A\exp(kn) \tag{9}$$

Here $T_\infty$, $A$, and $k$ are fitting parameters, of which only $T_\infty$, the asymptotic lifetime of the system, is of interest. The motivation to fitting an exponential function will become clear when discussing Fig. 3, but it is an obvious choice giving the decay rate of the Monte Carlo points. Figure 1 also plots the lifetimes computed with the standard Monte Carlo method (filled black circles), and the upper and lower bounds derived using, in each breaking step, appropriate means of the decay widths (the harmonic mean for the upper bound and the geometric bound for the lower bound [10]).

Two conclusions can be drawn from Fig. 1: (1) Both Monte Carlo methods give identical results (given the error bars); and (2) the Monte Carlo results are nicely delimited by the upper and lower bounds computed with the harmonic and geometric means. These results confirm the findings in Ref. [10]. It is worth stressing the coincidence between both Monte Carlo methods, as one method introduces randomness via quenched disorder (standard method) whereas the other introduces it via annealed disorder.

To check the quality of the exponential fit, a variable number of Monte Carlo data points were included in the fit. Figure 2 shows the evolution of the fitting parameter $T_\infty$ as a function of the number of Monte Carlo data points included in the fit. The horizontal axis in Fig. 2 gives the number of data points used in the exponential fit, starting at the right with 20 points (all available Monte Carlo data) and ending

on the left with 7 data points, those belonging to the biggest systems. Thus, in going from right to left in Fig. 2 we discard small-$N$ systems from the fit. As can be seen, after discarding systems smaller than $n=14$ (i.e., $N=2^{14}$ fibers) the value of $T_\infty$ reaches a plateau at $0.3328\pm0.0013$ (shaded area in Fig. 2).

Figure 3 looks at the same data from a different viewpoint. If lifetimes really follow an exponential decay to a nonzero value and we subtract this asymptotic value from the lifetime we will obtain a function that goes to zero as $n \to \infty$. Plotting the "corrected" lifetimes on a log-normal graph should give a straight line. This is a more sensitive test than the actual fit to Eq. (9), as deviations from a straight line are much easier to detect. We have subtracted from the Monte Carlo lifetimes a quantity $\epsilon$ and plot the resulting function on log-linear scales. Then, a least squares fit to the $n>14$ data points was performed and the correlation coefficient $r$ recorded. The parameter $\epsilon$ was changed until obtaining the highest correlation coefficient (i.e., the "best" straight line on log-normal scales). The obtained value, $\epsilon \equiv T_\infty = 0.3327\pm0.0001$, is compatible with the previous result $T_\infty = 0.3328\pm0.0013$. Both values are shown in Fig. 2 as a red line and a shaded area, respectively.

This paper has strengthened the view put forward in Ref. [10] that time-dependent HLS systems, in stark contrast to their static counterparts, do have a critical point, in the sense that their lifetime does not go to zero as the size of the system is increased. In fact the numerical results obtained in this paper are conclusive. In particular in Fig. 2 the horizontality of the last 8 points indicates that it is not necessary to go beyond the $n=20$ value studied here.

Table I summarizes what is known about the asymptotic properties of the different stress-transfer modalities of fiber bundles, both in the static and in the time-dependent versions. The result obtained for time-dependent HLS systems seems to break the symmetry suggested by previous findings and shown in Table I: for the ELS and LLS cases, when the static version has a critical point the time-dependent one also has it (ELS systems), or the other way around (LLS system). But for HLS systems the results presented here and in Refs. [3,5,10] strongly suggest that the time-dependent version has a critical point, notwithstanding the absence of critical point of the static version.

TABLE I. Main asymptotic results for the three standard modalities of fiber-bundle models in the static and time-dependent cases (see [10] and references herein).

| | ELS | LLS | HLS |
|---|---|---|---|
| Static | Critical point | No critical point | No critical point |
| Time-dep | Critical point | No critical point | Critical point |

From a practical point of view, when dealing with *finite* systems, the difference between having and not having a critical point is not so important as the rate at which a static HLS system loses strength is [3,12]

$$\sigma^* \propto \frac{1}{\ln(\ln N)},$$

where $\sigma^*$ is the stress on the system at failure. This is an extremely slow decrease in strength towards zero.

For many practical applications, the difference between "a very slow decrease towards zero" and "a decrease towards a nonzero value" is not fundamental. But from a *conceptual* point of view, the above difference is indeed *truly* fundamental.

## ACKNOWLEDGMENT

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